GALOIS MODULE STRUCTURE OF MILNOR K-THEORY MOD p^s IN CHARACTERISTIC p

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ABSTRACT. Let E be a cyclic extension of pth-power degree of a field F of characteristic p. For all m, $s \in \mathbb{N}$, we determine K_mE/p^sK_mE as a $(\mathbb{Z}/p^s\mathbb{Z})[\mathrm{Gal}(E/F)]$-module. We also provide examples of extensions for which all of the possible nonzero summands in the decomposition are indeed nonzero.

Let F be a field of characteristic p. Let K_mF denote the mth Milnor K-group of F and $k_mF = K_mF/pK_mF$. (See, for instance, [Mi] and [Ma, Chapter 14].) If E/F is a Galois extension of fields, let $G = \mathrm{Gal}(E/F)$ denote the associated Galois group. In [BLMS] the structure of k_mE as an $\mathbb{F}_pG$-module was determined when G is cyclic of pth-power order. In this paper we determine the Galois module structure of K_mE modulo p^s for $s \in \mathbb{N}$ and these same G. We also provide examples of extensions for which the possible free summands in the decomposition are all nonzero. These examples together with the results in [BLMS] show that the dimensions over $\mathbb{F}_p$ of indecomposable $\mathbb{F}_p[\mathrm{Gal}(E/F)]$-modules occuring as direct summands of k_mE are all powers of p and that all dimensions p^i, $i = 0, 1, \ldots, n$, indeed occur in suitable examples.

Recall the theorem of Bloch-Kato and Gabber (see [BK]): the sequence

$$0 \to k_mF \to \Omega_F^m \xrightarrow{\mathfrak{P}} \Omega_F^m/d\,\Omega_F^{m-1}$$

is exact, where Ω_F^m is the mth graded component of the exterior algebra on Kähler differentials and $\mathfrak{P}$ is the Artin-Schreier operator. In [I, §6], Izhboldin succeeded in providing an analogue of this important

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interpretation of k_mF, as follows: for $s \in \mathbb{N}$, the sequence

$$0 \to K_mF/p^sK_mF \xrightarrow{\delta} Q^m(F,s) \xrightarrow{\mathfrak{P}} Q^m(F,s)$$

is exact, where $Q^m(F,s)$ is the "Milnor-Witt" group of F, defined together with maps δ and $\mathfrak{P}$ in [I, §6]. These objects play an important role in the arithmetic of fields, higher class field theory, and Milnor K-theory of fields of characteristic p. (See, for instance, [FK] and the references therein.)

Since the classification problem of $(\mathbb{Z}/p^s\mathbb{Z})G$-modules for cyclic G is nontrivial and has not been completely solved—see, for instance, [T] for results and references—it is a pleasant surprise that the $(\mathbb{Z}/p^s\mathbb{Z})G$-modules K_mE/p^sK_mE have a simple description. The main ingredients we use to obtain this description are the lack of p-torsion in K_mE, due to Izhboldin [I], together with the result [BLMS] for the case $s = 1$—which also depends on Izhboldin's result.

Suppose that E/F is cyclic of degree p^n, and for $i = 0, 1, \ldots, n$, let E_i/F be the subextension of degree p^i of E/F and $G_i := \mathrm{Gal}(E_i/F)$. Set $R_s := \mathbb{Z}/p^s\mathbb{Z}$ and let $\mathbb{Z}_p$ be the ring of p-adic integers. We write $\iota_{F,E}\colon K_mF \to K_mE$ and $N_{E/F}\colon K_mE \to K_mF$ for the natural inclusion and norm maps, and we use the same notation for the induced maps between K_mF/p^sK_mF and K_mE/p^sK_mE.

Theorem. *There exists an isomorphism of R_sG-modules*

$$K_mE/p^sK_mE \simeq \oplus_{i=0}^{n} Y_i$$

where

- Y_n *is a free R_sG-module of rank* $\dim_{\mathbb{F}_p} N_{E/F}k_mE$,
- Y_i, $0 < i < n$, *is a free R_sG_i-module of rank*
$$\dim_{\mathbb{F}_p} N_{E_i/F}k_mE_i/N_{E_{i+1}/F}k_mE_{i+1},$$
- Y_0 *is a free R_s-module of rank*
$$\dim_{\mathbb{F}_p} k_mF/N_{E_1/F}k_mE_1 \textit{ with } Y_0^G = Y_0,$$
and
- *for each* $0 \le i \le n$, $Y_i \subseteq \iota_{E_i,E}(K_mE_i/p^sK_mE_i)$.

Moreover,

$$\widehat{K_mE} := \varprojlim_{s} K_mE/p^sK_mE \simeq \oplus_{i=0}^{n} \hat{Y}_i,$$

where each $\hat{Y}_i$ is a free $\mathbb{Z}_pG_i$-module of the same rank as Y_i.

1. Proof of the Theorem

We prove the result by induction on s. The case $s = 1$ is [BLMS, Theorem 2]. Assume therefore that $s > 1$ and the result holds for $s-1$:

$$K_mE/p^{s-1}K_mE = \oplus \tilde{Y}_i,$$

with each $\tilde{Y}_i$ a free $R_{s-1}G_i$-module $\tilde{Y}_i$ in the image of $\iota_{E_i,E}$, $0 \le i \le n$.

For each i with $0 \le i \le n$, let $B_{s-1,i} \subset i_{E_i,E}K_mE_i/p^{s-1}K_mE_i$ be an $R_{s-1}G_i$-base for the free $R_{s-1}G_i$-module $\tilde{Y}_i$. By induction the cardinality of $B_{s-1,i}$ is

$$|B_{s-1,i}| = \begin{cases} \dim_{\mathbb{F}_p} N_{E_i/F}k_mE_i/N_{E_{i+1}/F}k_mE_{i+1}, & i < n \\ \dim_{\mathbb{F}_p} N_{E/F}k_mE, & i = n. \end{cases}$$

Since the $\tilde{Y}_i$ are independent, the set

$$B_{s-1} := \cup_{0\le i\le n} B_{s-1,i} \subseteq K_mE/p^{s-1}K_mE$$

is $R_{s-1}G$-independent.

For each i, let $\mathcal{B}_i \subseteq \iota_{E_i,E}(K_mE_i)$ be a set of representatives for the elements of $B_{s-1,i}$, and let $B_{s,i} \subseteq K_mE/p^sK_mE$ be chosen to make the following first diagram commutative. The second diagram merely recalls where our $\mathcal{B}_i, B_{s,i}$ and $B_{s-1,i}$ are located.

$$\begin{tikzcd} \mathcal{B}_i \arrow[r, "\bmod p^s"] \arrow[d, "\bmod p^{s-1}"'] & B_{s,i} \arrow[dl, "\bmod p^{s-1}"] \\ B_{s-1,i} & \end{tikzcd} \qquad \begin{tikzcd} K_mE \arrow[r] \arrow[d] & K_mE/p^sK_mE \arrow[dl] \\ K_mE/p^{s-1}K_mE & \end{tikzcd}$$

Hence for each i we have bijections

$$\mathcal{B}_i \leftrightarrow B_{s,i} \leftrightarrow B_{s-1,i}$$

and $|B_{s,i}| = |B_{s-1,i}|$.

First we observe that every nonzero ideal V of R_sG_i contains p^{s-1} $(\tau-1)^{p^i}$, where τ is any fixed generator of G_i. Indeed consider $0 \ne \beta \in B$. By multiplying by an appropriate power of p, we may assume $0 \ne \beta \in p^{s-1}R_sG_i$. Let us write

$$\beta = \sum_{j=k}^{p^i-1} c_j(\tau-1)^j,$$

where each $c_j \in p^{s-1}R_s$, $j = k, \dots, p^i - 1$, and $c_k \notin p^s R_s = \{0\}$, say $c_k = p^{s-1}\tilde{c}_k$ with $\tilde{c}_k \notin pR_s$. Using the fact that $p^{s-1}(\tau - 1)^{p^i} = 0$ in R_sG_i we see that we can multiply β by $(\tau - 1)^{p^i-k-1}$ to obtain

$$0 \neq \tilde{c}_k p^{s-1}(\tau - 1)^{p^i-1} \in V.$$

Since $\tilde{c}_k \in U(R_s)$ we see that $p^{s-1}(\tau - 1)^{p^i-1} \in V$ as asserted.

Set Y_i to be the R_sG-submodule of K_mE/p^sK_mE generated by $B_{s,i}$. It is clear that $Y_i \subseteq \iota_{E_i,E}(K_mE_i/p^sK_mE_i)$ and hence Y_i is an R_sG_i-module.

Each element $b \in B_{s,i}$ generates in K_mE/p^sK_mE a free R_sG_i-module M_b, as follows. Suppose that M_b is not a free R_sG_i-module. Then the annihilator of b in R_sG_i is a nonzero ideal of R_sG_i. Let $\hat{b} \in \mathcal{B}_i$ and $\tilde{b} \in B_{s-1,i}$ correspond to b under the bijection above. Let also σ be a generator of G and $\bar{\sigma}$ its image in G_i. Since every nonzero ideal of R_sG_i contains $p^{s-1}(\bar{\sigma} - 1)^{p^i-1}$, for some $\hat{c} \in K_mE$ we have

$$p^{s-1}(\bar{\sigma} - 1)^{p^i-1}\hat{b} = p^s\hat{c}.$$

Since K_mE has no p-torsion [I, Theorem A], we obtain

$$p^{s-2}(\bar{\sigma} - 1)^{p^i-1}\hat{b} = p^{s-1}\hat{c}.$$

Then in $K_mE/p^{s-1}K_mE$

$$p^{s-2}(\bar{\sigma} - 1)^{p^i-1}\tilde{b} = 0,$$

contradicting the fact that $\tilde{b}$ lies in the $R_{s-1}G_i$-base $B_{s-1,i}$ for $\tilde{Y}_i$. (Alternatively we could use [T, Theorem 5.1] to show that M_b is a free R_sG_i-module.)

Now set $B_s := \cup_{0 \le i \le n} B_{s,i} \subseteq K_mE/p^sK_mE$. Suppose we have a relation

$$\sum r_\alpha b_\alpha = 0, \qquad b_\alpha \in B_s, \quad r_\alpha \in R_sG_\alpha,$$

such that $G_\alpha = G_i$ for a suitable i, $0 \le i \le n$, with $b_\alpha \in B_{s,i}$. Let $\tilde{b}_\alpha \in B_{s-1}$ correspond to b_α under the natural bijection, and similarly let $\tilde{r}_\alpha \in R_{s-1}G_\alpha$ be the image of $r_\alpha \in R_sG_\alpha$.

Working mod p^{s-1} we have $\sum \tilde{r}_\alpha \tilde{b}_\alpha = 0$. Since each $\tilde{b}_\alpha$ lies in the $R_{s-1}G$-independent set B_{s-1}, we deduce that $r_\alpha \in p^{s-1}R_sG_\alpha$ for each α. Write $r_\alpha = ps_\alpha$ for elements $s_\alpha \in R_sG_\alpha$. We rewrite the original relation as

$$\sum ps_\alpha b_\alpha = 0, \qquad b_\alpha \in B_s, \quad s_\alpha \in R_sG_\alpha.$$

Just as before we divide by p to obtain in $K_mE/p^{s-1}K_mE$

$$\sum \tilde{s}_\alpha \tilde{b}_\alpha = 0.$$

Again since each $\tilde{b}_\alpha \in B_{s-1}$, we deduce that $s_\alpha \in p^{s-1}R_sG_\alpha$. But then $r_\alpha = ps_\alpha = 0 \in R_sG_\alpha$ for each r_α, as desired.

Hence for each i in $0 \le i \le n$ we have that Y_i is a direct sum of free R_sG_i-modules M_b for $b \in B_{s,i}$, and moreover that $\sum Y_i = \oplus Y_i$. By Nakayama's Lemma, since $\mathcal{B}$ generates $K_m/p^{s-1}K_mE$ it also generates K_m/p^sK_mE, and hence $\oplus Y_i = K_mE/p^sK_mE$. (More explicitly, choose $\alpha \in K_mE/p^sK_mE$ and $\hat{\alpha} \in K_mE$ a lift of α. Since B_{s-1} spans $K_mE/p^{s-1}K_mE$ we have

$$\hat{\alpha} = \sum_{b \in \mathcal{B}} f_b b + p^{s-1}\hat{\gamma}$$

for some $\hat{\gamma} \in K_mE$, where each $f_b \in R_sG$ and all but finitely many $f_b = 0$. We also have $\hat{\gamma} = \sum g_b b + p^{s-1}\hat{\delta}$ for some $\hat{\delta} \in K_mE$, where again each $g_b \in R_sG$ and all but finitely many $g_b = 0$. Therefore

$$\alpha = \sum_{b \in \mathcal{B}} \left(f_b + p^{s-1}g_b\right) b$$

as elements of K_mE/p^sK_mE.)

The last statement concerning the equality $\widehat{K_mE} = \oplus_{i=0}^n \hat{Y}_i$ is obtained by passing to projective limits. $\square$

2. Examples of E/F with $Y_i \neq \{0\}$ for all i

Let p be an arbitrary prime number, and let q be an arbitrary prime number or 0. We show that for each $n, m \in \mathbb{N}$ there exists a cyclic field extension E/F of degree p^n and characteristic q such that for each i, $0 \le i < n$,

$$\dim_{\mathbb{F}_p} N_{E_i/F}k_mE_i/N_{E_{i+1}/F}k_mE_{i+1} \neq 0$$

and

$$\dim_{\mathbb{F}_p} N_{E/F}k_mE \neq 0.$$

Recall that we index E_i such that $F \subset E_i \subset E$ and $[E_i \colon F] = p^i$.

We are interested in the images of the quotient groups in k_mE. Because in the case $p = q$ the natural homomorphism $k_mF \to k_mE$ is injective, for the case of our main application we can work in k_mF.

2.1. **The case $m = 1$.**

Fix p, q, and n as above and set $m = 1$. We construct a field extension E/F as above together with elements

$$\begin{aligned} x_i &\in N_{E_i/F}(E_i^\times) \setminus N_{E_{i+1}/F}(E_{i+1}^\times)F^{\times p}, \quad 0 \le i < n, \\ x_n &\in N_{E/F}(E^\times) \setminus F^{\times p}. \end{aligned}$$

Let B be a field of characteristic q, and let A/B be a cyclic extension of degree p^n. Index the subfields A_i of A/B such that $[A_i : B] = p^i$, and denote by $\iota_{B,A_i} : K_1 B \hookrightarrow K_1 A_i$ the natural inclusion. Let σ be a generator of $\mathrm{Gal}(A/B)$ and set $\sigma_i = \sigma|_{A_i}$, the restricted map. Finally, assume that there exist elements $x_0, x_1, \ldots, x_n \in B^\times$ such that the following condition $(*)$ holds:

$$\begin{aligned} [\iota_{B,A_j}(x_j)]^{p^{n-j-1}} &\notin \langle [\iota_{B,A_j}(x_1)]^{p^{n-1}}, [\iota_{B,A_j}(x_2)]^{p^{n-2}}, \ldots, [\iota_{B,A_j}(x_n)] \rangle \\ &\subseteq A_j^\times / N_{A/A_j}(A^\times), \quad 0 \le j < n, \\ x_n &\notin B^{\times p}, \end{aligned}$$

where $[x]$ denotes the class of x and $\langle S \rangle$ the subgroup generated by a set S in the named factor group. At the end of this section we shall create an example where condition $(*)$ holds.

Now consider cyclic algebras

$$\mathcal{A}_j = (A/B, \sigma, x_j^{p^{n-j}}), \quad 1 \le j \le n.$$

Observe that

$$[\mathcal{A}_j] = [(A_j/B, \sigma_j, x_j)] \in \mathrm{Br}(B), \quad 1 \le j \le n$$

([P, Chapter 15, Corollary b]), where $\mathrm{Br}(B)$ denotes the Brauer group of B. Let F be the function field of the product of Brauer-Severi varieties of $\mathcal{A}_1, \ldots, \mathcal{A}_n$. (See [SV, page 735]; see also [J, Chapter 3] for basic properties of Brauer-Severi varieties.)

Let E be the compositum $A \cdot F$ of the fields F and A. Since F is a regular extension of B, we see that E/F is a cyclic extension of degree p^n. We denote again as σ the generator of $\mathrm{Gal}(E/F)$ which restricts to $\sigma \in \mathrm{Gal}(A/B)$, and we write $E_k = A_k \cdot F$ for $k = 0, 1, \ldots, n$. Now $[\mathcal{A}_j \otimes_B F] = 0 \in \mathrm{Br}(F)$, $j = 1, \ldots, n$, because F splits each $\mathcal{A}_j$. Hence

$$0 = [(E/F, \sigma, x_j^{p^{n-j}})] = [(E_j/F, \sigma_j, x_j)],$$

and so $x_j \in N_{E_j/F}(E_j^\times)$ as desired (see [P, Chapter 15, page 278]).

However, we claim that

$$\begin{aligned} &x_j \notin (N_{E_{j+1}/F}(E_{j+1}^\times))F^{\times p}, \quad 0 \le j < n, \\ &x_n \notin F^{\times p}. \end{aligned}$$

Since $x_n \notin B^{\times p}$ by hypothesis and F/B is a regular extension, we have $x_n \notin F^{\times p}$. Assume then that $0 \le j < n$ and, contrary to our statement,

$$x_j \in (N_{E_{j+1}/F}(E_{j+1}^\times))F^{\times p}.$$

Then we have $x_j f^p \in N_{E_{j+1}/F}(E_{j+1}^\times)$ for some $f \in F^\times$. Hence

$$[(E_{j+1}/F, \sigma_{j+1}, x_j f^p)] = 0 \in \mathrm{Br}(F)$$

and so

$$\begin{aligned} [(E_{j+1}/F, \sigma_{j+1}, x_j)] &= -[(E_{j+1}/F, \sigma_{j+1}, f^p)] \\ &= -[(E_j/F, \sigma_j, f)]. \end{aligned}$$

(In the case $j = 0$, we use $(E_0/F, \sigma_0, f)$ to denote the zero element in $\mathrm{Br}(F)$.) Consequently $(E_{j+1}/F, \sigma_{j+1}, x_j)$ is split by E_j. (See [P, Chapter 15, Proposition b].)

But then

$$[(E_{j+1}/E_j, \sigma_{j+1}^{p^j}, \iota_{F,E_j}(x_j))] = 0 \in \mathrm{Br}(E_j).$$

(See [D, page 74].) Hence $[(E/E_j, \sigma^{p^j}, \iota_{F,E_j}(x_j^{p^{n-j-1}}))] = 0 \in \mathrm{Br}(E_j)$. But $E_j = A_j \cdot F$ is the function field of the product of the Brauer-Severi varieties of $\mathcal{A}_k \otimes_B A_j$ for $k = 1, \dots, n$. Therefore

$$[(A/A_j, \sigma^{p^j}, \iota_{B,A_j}(x_j^{p^{n-j-1}}))] \in \langle [\mathcal{A}_k \otimes_B A_j], k = 1, \dots, n \rangle \subseteq \mathrm{Br}(A_j).$$

Consequently

$$\begin{aligned} [\iota_{B,A_j}(x_j^{p^{n-j-1}})] &\in \langle [\iota_{B,A_j}(x_1)]^{p^{n-1}}, \dots, [\iota_{B,A_j}(x_n)] \rangle \\ &\in A_j^\times / N_{A/A_j}(A^\times), \end{aligned}$$

a contradiction to condition $(*)$.

Thus we have shown that a required extension E/F exists with elements $x_0, x_1, \dots, x_n$, provided that we can produce a field extension A/B and elements $x_0, x_1, \dots, x_n \in B^\times$ such that condition $(*)$ is valid. Now we show that such an extension and elements exist.

Let $B := C(x_0, x_1, \dots, x_n)$, where C is a field with characteristic q and $x_0, x_1, \dots, x_n$ are algebraically independent elements over C. Assume also that there exists a cyclic extension D/C of degree p^n with

Galois group $G = \langle\sigma\rangle$. Finally, let $A := D(x_0, x_1, \ldots, x_n)$. Thus A/B is a cyclic extension of degree p^n.

We claim that condition $(*)$ holds. Clearly $x_n \notin B^{\times p}$. Contrary to our claim assume that

$$x_j^{p^{n-j-1}} = x_1^{c_1 p^{n-1}} \cdots x_n^{c_n} N_{A/A_j}(\gamma)$$

where $c_i \in \mathbb{Z}$, $\gamma \in A^\times$, and $0 \leq j < n$. Write

$$\gamma = u \cdot \frac{p_1 \cdots p_t}{q_1 \cdots q_w},$$

where $\{p_1, \ldots, p_t\}$ and $\{q_1, \ldots, q_w\}$ are disjoint sets of primes in the ring $D[x_0, x_1, \ldots, x_n]$ and $u \in U(D[x_0, x_1, \ldots, x_n]) = D^\times$.

We write $H_j = \mathrm{Gal}(A/A_j)$ for $j = 0, 1, \ldots, n-1$. Then we have

$$x_j^{p^{n-j-1}} \prod_{h \in H_j} \prod_{i=1}^{w} h(q_i) = x_1^{c_1 p^{n-1}} \cdots x_n^{c_n} N_{A/A_j}(u) \prod_{h \in H_j} \prod_{i=1}^{t} h(p_i).$$

In order to show that this equation is impossible, consider the discrete valuation v_j on A such that $v_j(x_j) = 1$, $v_j(x_k) = 0$ if $j \neq k$, and $v_j(d) = 0$ for $d \in D$. Consider first the case when $x_j \nmid q_l$, $l = 1, \ldots, w$. Then the value of v_j on the left-hand side is p^{n-j-1}, while the value of v_j on the right-hand side is at least p^{n-j}. Indeed if $c_j \neq 0$ this is true as $x_j^{p^{n-j}}$ divides the right-hand side of our equation. If $c_j = 0$ or $j = 0$ (in which case c_0 is not defined) then since x_j divides the left-hand side we see that there exists p_l, $l \in \{1, \ldots, t\}$, such that $x_j \mid p_l$. Because $|H_j| = p^{n-j}$ we see again that the value of v_j on the right-hand side is again at least p^{n-j}. Thus $x_j \mid q_l$ for some $l = 1, \ldots, w$, and so $x_j \nmid p_l$ for any $l = 1, \ldots, t$. Then the value of v_j on the left-hand side of our equation is $p^{n-j-1} + p^{n-j}$ while the value of v_j on the right-hand side is at most p^{n-j}. Thus we see that the equation above is impossible. Hence condition $(*)$ is valid and we have established the desired example of E/F in the case $m = 1$.

2.2. **The case $m > 1$.**

Fix m, n, and q as above and let L/K be a field extension satisfying the case $m = 1$ with the same n and q. Let $x_0, x_1, \ldots, x_n \in K^\times$ such that

$$\begin{aligned} &x_i \in N_{L_i/K}(L_i^\times) \setminus N_{L_{i+1}/K}(L_{i+1}^\times) K^{\times p}, \quad 0 \leq i < n, \\ &x_n \in N_{L/K}(L^\times) \setminus K^{\times p}. \end{aligned}$$

Consider the field of the iterated power series $F := K((y_1))\cdots((y_{m-1}))$. Then $E := L \cdot F$ is a cyclic extension of degree p^n over F. For each $j \in \{0, 1, \dots, n\}$ consider the element

$$\alpha_j = \{x_j, y_1, \dots, y_{m-1}\} \in k_m F.$$

(If $m = 1$ then $\alpha_j = \{x_j\}$.) By our hypothesis and the projection formula for the norm map in K-theory, we have

$$\alpha_j \in N_{E_j/F} k_m E_j.$$

Now for $0 \le j < n$ we shall prove by induction on $m \in \mathbb{N}$ that

$$\alpha_j \notin N_{E_{j+1}/F} k_m E_{j+1}.$$

If $m = 1$ then our statement is true by the choice of the field extension L/K and the elements x_j. Assume then that $m > 1$ and that our statement is true for $m - 1$.

Consider the complete discrete valuation v on F with uniformizer y_{m-1} and residue field $F_v = K((y_1))\cdots((y_{m-2}))$ if $m > 2$ and $F_v = K$ if $m = 2$.

For the sake of simplicity we denote by E' the field E_{j+1}, and denote the unique extension of v on E' again by v. Since we are considering an unramified extension we assume that both valuations are normalized. Let $\partial : k_m F \to k_{m-1} F_v$ and $k_m E' \to k_{m-1} E'_v$ be the homomorphisms induced by residue maps in Milnor K-theory. Then applying [K, Lemma 3] we see that the following diagram is commutative:

$$\begin{array}{ccc} k_m E' & \xrightarrow{\partial} & k_{m-1} E'_v \\ {\scriptstyle N_{E'/F}} \downarrow & & \downarrow {\scriptstyle N_{E'_v/F_v}} \\ k_m F & \xrightarrow{\partial} & k_{m-1} F_v. \end{array}$$

If $\alpha_j \in N_{E'/F} k_m E'$, then $\partial \alpha_j \in N_{E'_v/F_v} k_{m-1} E'_v$.

But $\partial \alpha_j = \{x_j, y_1, \dots, y_{m-2}\}$ if $m > 2$ and $\partial \alpha_j = \{x_j\}$ if $m = 2$, a contradiction in either case. Therefore we have constructed a field extension E/F with the desired properties.

Remarks. In [MSS1] we determined the $\mathbb{F}_p G$-module structure of $k_1 E$ for all cyclic extensions E/F of degree p^n, where G is the Galois group. In particular, the decomposition does not depend upon the characteristic of the base field. The ranks of the free $\mathbb{F}_p G_i$-summands appearing in that decomposition are again determined by the image of $N_{E_i/F}(E_i^\times)/N_{E_{i+1}/F}(E_{i+1}^\times)$ in $E^\times/E^{\times p}$.

When no primitive pth root of unity lies in F, we have that $F^\times/F^{\times p}$ embeds in $E^\times/E^{\times p}$. Therefore the construction given above for field extensions E/F applies in this case, and these free $\mathbb{F}_pG_i$-summands do indeed occur for any characteristic. When a primitive pth root of unity is in $F^\times$, the kernel of the homomorphism $F^\times/F^{\times p} \to E^\times/E^{\times p}$ is generated by a class $[a] \in F^\times/F^{\times p}$, where $E_1 = F(\sqrt[p]{a})$. In this case it is enough to additionally require that $a \in N_{E/F}(E^\times)$ when $p = 2$ and $n = 1$ (since the condition is automatic otherwise), and the construction above of field extensions E/F applies again.

If F contains a primitive pth-root of unity and $m = 1$, the decomposition contains at most one other indecomposable module, a cyclic $\mathbb{F}_pG$-module of dimension $p^k + 1$ for $k \in \{-\infty, 0, 1, \dots, n-1\}$ (where we set $p^{-\infty} = 0$). In [MSS2] we showed that all of these modules are realizable as well.

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